Tutorial 4

In the following problems, V denotes a finite-dimensional inner product space.

1. Suppose $\mathbb{F} = \mathbb{R}$.
   (a) Show that every self-adjoint $T \in \mathcal{L}(V)$ has a cube root, i.e. an operator $S \in \mathcal{L}(V)$ such that $S^3 = T$.
   (b) Find a linear operator $T \in \mathcal{L}(V)$ such that T does not have a square root, i.e. an operator $S \in \mathcal{L}(V)$ such that $T = S^2$.
2. Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$ is self-adjoint with eigenvalues $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and corresponding eigenspaces $E_{\lambda_1}, \ldots, E_{\lambda_k}$. Prove there exists $a_1, \ldots, a_k \in \mathbb{R}$ such that
$$T = a_1 P_{E_{\lambda_1}} + \cdots + a_k P_{E_{\lambda_k}}$$
3. Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$ is self-adjoint. Show that for all $c \in \mathbb{R}$ nonzero, $T + ciI$ is invertible.
4. Let $T \in \mathcal{L}(V)$ be self-adjoint and $v_1, v_2 \in V$ be eigenvectors of T with distinct eigenvalues λ_1 and λ_2, respectively.
   (a) Show that $\lambda_1, \lambda_2 \in \mathbb{R}$.
   (b) Show that $\langle v_1, v_2 \rangle = 0$. What can you deduce about the composition $P_{E_{\lambda_1}} P_{E_{\lambda_2}}$, where E_{λ_i} is the eigenspace corresponding to λ_i?
5. Let $T \in \mathcal{L}(V)$ be normal. Recall last week that we proved null $T^* =$ null T and range $T^* =$ range T.
   (a) Show that null $T^2 =$ null T.
   (b) Show that range $T^2 =$ range T.
6. We say a self-adjoint operator $T \in \mathcal{L}(V)$ is *positive-definite* if for all $v \in V$ nonzero, $\langle T(v), v \rangle > 0$. Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$ is positive-definite.
   (a) Show that T is invertible and T^{-1} is positive-definite.
   (b) Show that $\langle u, v \rangle_T = \langle T(u), v \rangle$ defines an inner product on V.
   (c) Suppose $S \in \mathcal{L}(V)$ is self-adjoint. Prove that ST and TS are diagonalizable.